

A GÖDEL–SCOTT THEOREM OF NECESSARY EXISTENCE

OPENAI

Does God exists?

We formulate the question as an existence problem in a specified higher-order modal theory. The resulting theorem is affirmative: the Gödel–Scott axioms imply necessary realization of a unique God-like individual. The proof below is followed by its algebraic and proof-theoretic consequences, and a probability-logic formulation.

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1. THE THEOREM AND ITS SETTING.

1.1. The mathematical claim. Let Γ denote the axiom schemas and definitions in §2, imposed globally on an S5 frame. The property G is the defined God-like predicate of that theory. We prove

$$(1.1) \quad \Gamma \models_{S5} \Box \exists! x G(x).$$

Since S5 frames are reflexive, (1.1) implies $\exists! x G(x)$ at the distinguished actual world of every model. This is a theorem of mathematical logic. Its probability formulation is a separate corollary of entailment, not an additional assumption in the existence proof.

The framework is specifically the consistent *Scott amendment* of Gödel’s ontological schema. The possession condition in the definition of essence is retained. The initial formalization [7, 3, 11] and the later analysis by Benz Müller and Scott [6, 12] provide the central reference points. Alternative formulations are considered only in the survey; they are not combined with the main axioms.

1.2. Interpretation and proof obligation. An interpretation of G as a creator or an omniscient personal being is not a primitive theorem of logic. Here the word God-like is fixed by Definition (2.3). Thus the affirmative theorem has a determinate mathematical object and a determinate class of structures. An assertion about the actual universe additionally requires identifying its intended interpretation with a model of these axioms.

The equation $\Pr(H \mid \Gamma) = 1$ does not imply $\Pr(H) = 1$ in an unspecified probability space. Section 7 gives the precise probability-space formulation.

1.3. Organization. The proof proceeds from possible realization to essence, then from essence to necessary realization. Uniqueness and a principal-ultrafilter description follow from the same assumptions. We next exhibit models, characterize the full-property case, and explain why omitting the essence conjunct yields inconsistency. Incompleteness concerns the internal proof theory of effective arithmetic extensions; probability logic then translates semantic entailment into a universal conditional-certainty assertion. The research survey follows those same dependencies.

2. LANGUAGE, SEMANTICS, AND AXIOMS.

2.1. Worlds and properties. Fix nonempty sets W and D , a distinguished world $w_0 \in W$, and an equivalence relation $R \subseteq W \times W$. Individuals belong to the constant domain D . An intensional unary property is a function

$$\varphi : W \times D \longrightarrow \{0, 1\}.$$

We initially use full property quantification. Consequently all complements, conjunctions, identity properties, and world-dependent properties used below are available. Write $\varphi(x)$ for evaluation at the current world. Identity on D is rigid.

The modal operators have the usual Kripke interpretation:

$$(2.1) \quad \mathcal{M}, w \models \Box A \iff \forall v (wRv \Rightarrow \mathcal{M}, v \models A),$$

$$(2.2) \quad \mathcal{M}, w \models \Diamond A \iff \exists v (wRv \wedge \mathcal{M}, v \models A).$$

A universally valid schema holds at every world for every admissible assignment. Property variables retain their intensions when moving between worlds. This prevents an implicit switch between a property and its extension at one world.

Lemma 2.1. *Every S5 frame validates*

$$\Diamond \Box A \rightarrow \Box A, \quad \Box A \rightarrow A.$$

Proof. If wRv , the equivalence classes of w and v coincide. Thus, if A holds at all worlds accessible from such a v , it holds at all worlds accessible from w . Reflexivity gives the second implication. $\square$

These frame conditions are sufficient for the present derivation; no minimality claim is made. Published work also verifies a Scott-style theorem in KB [11]. General-model semantics and weaker property comprehensions are discussed in §8.

2.2. The Gödel–Scott definitions. Let $P(\varphi)$ be a primitive positivity predicate, potentially evaluated at each world. It is an abstract mathematical predicate, not a numerical sign or a moral judgment. Define

$$(2.3) \quad G(x) \iff \forall \varphi [P(\varphi) \rightarrow \varphi(x)],$$

$$(2.4) \quad \text{Ess}(\varphi, x) \iff \varphi(x) \wedge \forall \psi [\psi(x) \rightarrow \Box \forall y (\varphi(y) \rightarrow \psi(y))],$$

$$(2.5) \quad \text{NE}(x) \iff \forall \varphi [\text{Ess}(\varphi, x) \rightarrow \Box \exists y \varphi(y)].$$

The definitions are global. In (2.4) the conjunct $\varphi(x)$ is essential to the precise system being studied. Equation (2.5) defines necessary existence through necessary exemplification of essences; we use that equation rather than infer existence from the name of the predicate.

2.3. The axioms. For $\neg\varphi := \lambda x. \neg\varphi(x)$, impose

- (A1) $P(\neg\varphi) \iff \neg P(\varphi),$
- (A2) $P(\varphi) \wedge \Box\forall x(\varphi(x) \rightarrow \psi(x)) \rightarrow P(\psi),$
- (A3) $P(G),$
- (A4) $P(\varphi) \rightarrow \Box P(\varphi),$
- (A5) $P(NE).$

All free property variables are universally quantified. All five schemas hold at every world. Globality is needed when the existence argument moves to an accessible world.

A1 chooses exactly one property from each complementary pair. A2 closes positivity under necessary implication. A3 declares the joint God-like property positive. A4 makes positive status persistent over accessibility. A5 declares the defined necessary-existence property positive. These are the hypotheses of the existence theorem.

3. THE EXISTENCE PROOF.

3.1. Possible realization.

Lemma 3.1. *At every world, $P(\varphi) \rightarrow \Diamond\exists x\varphi(x)$.*

Proof. Assume $P(\varphi)$ but $\neg\Diamond\exists x\varphi(x)$. Then $\Box\forall x\neg\varphi(x)$, hence

$$\Box\forall x(\varphi(x) \rightarrow \neg\varphi(x)).$$

A2 yields $P(\neg\varphi)$ and A1 yields $\neg P(\varphi)$, a contradiction. $\square$

Corollary 3.2. *$\Diamond\exists xG(x)$ holds at every world.*

Proof. Apply Lemma 3.1 to A3. $\square$

The corollary does not infer joint possibility from separate possibilities. That inference would be invalid: a property and its complement can be separately realizable without being jointly realizable. The joint property appears explicitly in A3.

3.2. Essence.

Lemma 3.3. *If $G(a)$, then for every ψ ,*

$$P(\psi) \iff \psi(a).$$

Proof. The forward implication is Definition (2.3). Conversely, if $\psi(a)$ and $\neg P(\psi)$, A1 gives $P(\neg\psi)$; Definition (2.3) gives $\neg\psi(a)$, a contradiction. $\square$

Lemma 3.4. $G(a) \rightarrow \text{Ess}(G, a).$

Proof. Assume $G(a)$ at w . For any ψ with $\psi(a)$ there, Lemma 3.3 yields $P(\psi)$. A4 yields $\Box P(\psi)$. At each accessible world, Definition (2.3) gives $\forall y(G(y) \rightarrow \psi(y))$. Thus the second conjunct of (2.4) holds for $\varphi = G$, and the first conjunct is the initial assumption. $\square$

3.3. Necessary realization.

Lemma 3.5. *At every world, $\exists xG(x) \rightarrow \Box\exists xG(x)$.*

Proof. Choose a with $G(a)$. A5 and (2.3) give $NE(a)$. Lemma 3.4 gives $Ess(G, a)$. Definition (2.5) therefore gives $\Box\exists yG(y)$. $\square$

Theorem 3.6 (Gödel–Scott necessary existence). *Every model of Γ satisfies $\Box\exists xG(x)$ at every world. It therefore satisfies $\exists xG(x)$ at its distinguished actual world.*

Proof. Corollary 3.2 supplies an accessible world containing a God-like individual. At that world Lemma 3.5 gives $\Box\exists xG(x)$. Hence the original world satisfies $\Diamond\Box\exists xG(x)$. Lemma 2.1 yields necessity and then actual-world realization. $\square$

Theorem 3.7 (Uniqueness). $\Gamma \models \Box\exists!xG(x)$.

Proof. At any world suppose $G(a)$ and $G(b)$. The identity property $I_a(x) := (x = a)$ is admissible. Since $I_a(a)$ holds, Lemma 3.3 gives $P(I_a)$. Definition (2.3) gives $I_a(b)$, so $b = a$. Apply this at each accessible world and use Theorem 3.6. $\square$

Theorems 3.6 and 3.7 establish the main claim. The following sections develop its algebraic, model-theoretic, and probabilistic consequences.

4. ALGEBRAIC STRUCTURE AND THE ROLE OF INFINITY.

4.1. Positive properties as evaluation. Fix a world w and let g_w be its unique God-like individual. By Lemma 3.3,

$$(4.1) \quad P_w(\varphi) \iff \varphi(w, g_w) = 1.$$

Consequently the positive properties form a principal ultrafilter on the Boolean algebra of all subsets of $W \times D$, with principal point (w, g_w) .

Proposition 4.1. *Write $U_w = \{\varphi : P_w(\varphi)\}$. Then $\top \in U_w$, $\perp \notin U_w$, U_w is upward closed for pointwise implication and closed under finite conjunction, and exactly one of $\varphi, \neg\varphi$ belongs to U_w .*

Proof. By (4.1), membership means truth at the point (w, g_w) . Each assertion follows by the corresponding Boolean truth rule at that point. For example, if $\varphi, \psi \in U_w$, both are true there, so $\varphi \wedge \psi \in U_w$. $\square$

Proposition 4.2. *In full property semantics,*

$$\bigcap_{\varphi \in U_w} \{(v, x) : \varphi(v, x) = 1\} = \{(w, g_w)\}.$$

Proof. Every member of U_w contains (w, g_w) by (4.1). Conversely, the singleton property of (w, g_w) is available under full property quantification and belongs to U_w by the same equation. The intersection is therefore no larger than that singleton. $\square$

These are consequences of the modal theorem, not a replacement of its modal premises by bare ultrafilter existence. The distinction matters: a principal ultrafilter alone says nothing about how its generator varies with the world. Research on modal ultrafilters makes this distinction explicit [96, 98, 101].

4.2. Why a general ultrafilter is insufficient.

Definition 4.3. An ultrafilter U on a set X is a proper filter on $\mathcal{P}(X)$ which, for every $A \subseteq X$, contains exactly one of A and $X \setminus A$. It is principal if $U = \{A : x \in A\}$ for some $x \in X$.

Proposition 4.4. *Every ultrafilter on a finite nonempty set is principal.*

Proof. If no singleton belongs to U , the complement of every singleton belongs to U . Intersect these finitely many complements. Their intersection is empty, contradicting propriety of the filter. Thus $\{x\} \in U$ for some x . Upward closure and propriety show $A \in U$ exactly when $x \in A$. $\square$

Proposition 4.5. *Assuming the ultrafilter lemma, there is an ultrafilter on $\mathbb{N}$ with empty intersection of all its members.*

Proof. Extend the cofinite filter to an ultrafilter U . For each n , the set $\mathbb{N} \setminus \{n\}$ is cofinite and belongs to U . Thus no n belongs to the intersection. Moreover, U contains no finite set: intersecting such a set with its cofinite complement would give the empty set. $\square$

Finite conjunction closure therefore does not imply simultaneous realization of an arbitrary infinite family.

4.3. Stone representation and what it represents. Let B be a Boolean algebra and $\text{Ult}(B)$ its set of ultrafilters. Define

$$\hat{b} = \{U \in \text{Ult}(B) : b \in U\}.$$

Under the Boolean prime ideal theorem, the map $b \mapsto \hat{b}$ is an injective Boolean homomorphism, the algebraic core of Stone representation [104].

Proof of the stated representation fact. The Boolean operations are preserved by the ultrafilter laws. If $b \neq c$, then their symmetric difference d is nonzero. Extend the proper filter generated by d to an ultrafilter U . Then $d \in U$, so exactly one of b, c belongs to U . Hence $\hat{b} \neq \hat{c}$. $\square$

Representation supplies points in a space of ultrafilters. It does not identify them with individuals in an independently prescribed physical domain. In the Gödel–Scott theorem, (4.1) supplies the particular evaluation point only after the modal existence proof has been completed.

5. MODELS, COLLAPSE, AND THE ESSENCE AMENDMENT.

5.1. A nonvacuity witness.

Proposition 5.1. Γ has a model.

Proof. Take $W = \{w\}$, $D = \{d\}$ and $R = \{(w, w)\}$. Set $\mathbf{P}(\varphi)$ iff $\varphi(w, d) = 1$. A1, A2 and A4 follow immediately. Every positive property holds of d , so $\mathbf{G}(d)$ and A3 hold. Every essence of d holds of d by (2.4), and is consequently exemplified at the sole accessible world. Thus $\mathbf{NE}(d)$ and A5 hold. $\square$

The existence result is therefore not justified merely by explosion from an inconsistent theory. For a sound proof calculus, Proposition 5.1 establishes consistency externally in the surrounding metatheory.

5.2. Modal collapse.

Proposition 5.2. *For every proposition p for which $\lambda x.p$ is an admissible property,*

$$\Gamma \models p \rightarrow \Box p.$$

Proof. At a world where p holds, choose a with $G(a)$ by Theorem 3.6. For $\psi(x) := p$, we have $\psi(a)$. Lemma 3.4 gives $\Box \forall y (G(y) \rightarrow p)$. Theorem 3.6 also gives $\Box \exists y G(y)$. Together they imply $\Box p$. $\square$

Modal collapse is a consequence of the stated axioms and distinguishes this theory from collapse-free variants [102, 96, 31, 27].

5.3. Classification under full property quantification.

Theorem 5.3. *In the full-property semantics of §2, a structure satisfies Γ if and only if R is the identity relation and there is a function $g : W \rightarrow D$ such that*

$$(5.1) \quad P_w(\varphi) \iff \varphi(w, g(w)).$$

The defined predicates are interpreted by (2.3)–(2.5).

Proof. Suppose first that Γ holds. Uniqueness defines $g(w)$ without choosing between multiple witnesses. Equation (5.1) is Lemma 3.3. If wRv and $w \neq v$, use the admissible property $\psi(u, x) := (u = w)$. It holds of $g(w)$ at w . Lemma 3.4 implies that every God-like individual at v has ψ there. Theorem 3.6 supplies such an individual, but $\psi(v, x)$ is false. This contradiction excludes distinct accessible worlds. Reflexivity then makes R the identity.

Conversely, let R be the identity and define positivity by (5.1). A1 and A2 follow from evaluation at $(w, g(w))$, and A4 holds since the only accessible world is w . Definition (2.3) gives $G(w, x)$ iff $x = g(w)$: the forward direction uses the identity property of $g(w)$ and the reverse direction uses evaluation. Thus A3 holds. For any x , every essence of x holds of x at w , so it has a witness in the only accessible world. Hence $NE(w, x)$ holds for all x , establishing A5. $\square$

The classification is specific to full property quantification, particularly the world-separating properties used in its proof. For a restricted semantic language, truth-indiscernible worlds need not be identical. The theorem is not asserted for every Gödelian variant or every Henkin property domain.

Corollary 5.4. *On fixed finite carriers with $|W| = k \geq 1$ and $|D| = n \geq 1$, the full-property S5 models of Γ have identity accessibility and exactly n^k possible positivity interpretations.*

Proof. By Theorem 5.3, models correspond exactly to functions $g : W \rightarrow D$. There are n^k such functions. Distinct functions give different positivity predicates, as identity properties distinguish their values. $\square$

This further shows why $\Box \exists! x G(x)$ should not be read as a claim that the same individual is God-like in every world of a disconnected frame. A function g can take different values in different components. The theorem's necessity ranges over accessibility.

5.4. Why the unamended essence definition fails here. Replace (2.4) by

$$\text{Ess}^-(\varphi, x) \iff \forall \psi [\psi(x) \rightarrow \Box \forall y (\varphi(y) \rightarrow \psi(y))]$$

and define NE^- by using Ess^- in (2.5).

Proposition 5.5. *In the same S5 setting, A1, A2 and $P(NE^-)$ are inconsistent with these unamended definitions.*

Proof. The false property $\perp$ necessarily implies every property, so $\text{Ess}^-(\perp, x)$ holds for every x . Therefore $\text{NE}^-(x)$ would imply $\Box\exists y\perp$, which is false on a reflexive frame. Thus NE^- has no possible instance. But the proof of Lemma 3.1, using only A1 and A2, shows that a positive property must have a possible instance. Apply it to $\text{P}(\text{NE}^-)$ for a contradiction. $\square$

This explains why the corrected Scott framework is specified from the outset. The research literature supplies more general analyses of the inconsistency [4, 2]; the argument above proves precisely the S5 claim needed here.

6. PROOF THEORY, COMPACTNESS, AND INCOMPLETENESS.

6.1. An existence theorem is not an internal consistency oracle. Let T be a consistent, effectively axiomatized classical theory extending sufficient arithmetic for the standard incompleteness theorems. If the modal proof is implemented in T , distinguish the encoded modal object theory from its arithmetic metatheory. A claim of full higher-order semantic completeness is not part of this implementation assumption.

The first incompleteness theorem, in Rosser’s strengthening, supplies an arithmetic sentence σ such that

$$(6.1) \quad T \not\vdash \sigma, \quad T \not\vdash \neg\sigma.$$

The second theorem, with standard arithmetized provability and sufficient arithmetic, yields

$$(6.2) \quad T \not\vdash \text{Con}(T).$$

The original and formalized sources are [137, 134, 130, 139]. The hypotheses are essential: these are theorems about effective theories with arithmetic strength, not every collection of mathematical truths.

Proposition 6.1. *If $T \vdash H$ and T satisfies the hypotheses of (6.2), then $T \not\vdash H \rightarrow \text{Con}(T)$.*

Proof. Otherwise modus ponens would give $T \vdash \text{Con}(T)$, contradicting (6.2). $\square$

Thus a theory can prove its God-like existence sentence while remaining incomplete. Conversely, incompleteness does not supply a God-like existence sentence. A model construction in a stronger metatheory can establish consistency of a weaker theory without contradicting (6.2); it is not that weaker theory’s own internal consistency proof.

6.2. Löb’s theorem and reflection. With the standard Hilbert–Bernays–Löb derivability conditions, Löb’s theorem states

$$T \vdash \text{Prov}_T(\ulcorner A \urcorner) \rightarrow A \quad \implies \quad T \vdash A.$$

A formal development is included in [139]. Taking $A = \perp$ recovers the obstruction to a consistent T proving its standard consistency sentence.

6.3. First-order compactness and witness construction. The first-order compactness theorem states that a theory is satisfiable if each finite subset is satisfiable. It can be proved from completeness: an unsatisfiable theory proves contradiction, and a finite proof uses only finitely many premises. Foundational and formalized treatments appear in [125, 127, 119, 122, 117].

For a family of properties $\{P_i : i \in I\}$ in a first-order language, add a constant c . If every finite subset of

$$B \cup \{P_i(c) : i \in I\}$$

is satisfiable, compactness supplies some model realizing the entire family. It does not guarantee a witness in a prescribed model. In the standard natural numbers the requirements $c > n$ for all

standard n are finitely satisfiable but have no joint witness. A compactness model of arithmetic may instead contain a nonstandard element.

This directly explains two restrictions in the ontological proof. First, individual realizability cannot replace finite joint realizability. Second, existence in some mathematical structure does not identify that structure with the actual universe. Full second-order semantics also cannot inherit first-order compactness without justification. Henkin's general-model treatment of type theory [120, 110, 58, 113] concerns a different semantic completeness claim.

6.4. A separation theorem.

Proposition 6.2. *If T_0 is any satisfiable first-order theory and U is a fresh unary predicate with no connecting axioms, then T_0 entails neither $\exists xU(x)$ nor its negation.*

Proof. Take $\mathcal{M} \models T_0$ with nonempty domain M . Interpret U as $\emptyset$ in one expansion and as M in another. All sentences of T_0 retain their truth values, while the existential assertion is respectively false and true. $\square$

The construction applies to any satisfiable theory in the original language. An existence assertion about the new predicate requires connecting axioms. The Gödel–Scott definitions and axioms provide such structure explicitly.

6.5. Material implication and explosion. Put $H := \exists xG(x)$ and let Q be an uninterpreted predicate. Once Theorem 3.6 is established,

$$(6.3) \quad \Gamma \models \neg H \rightarrow \forall aQ(a)$$

is a truth-functional consequence for every interpretation of Q . Hence the sentence “If there is no God, everything is permitted” can be represented by (6.3), but it is not an independent mathematical input. No moral properties of Q are assumed.

Likewise, classical explosion says $A, \neg A \vdash B$ for every B : infer $A \vee B$ and use $\neg A$. Explosion concerns contradiction, not absence of a deity. Proposition 5.1 ensures that the main theorem is not merely an application of explosion. This is the relevant formal connection among the permission sentence, consistency, and the existence proof.

7. FROM ENTAILMENT TO PROBABILITY.

7.1. Certainty conditional on the model class. Let Ω be a nonempty set of pointed models of Γ , and equip it with any probability measure μ . Write

$$E_H = \{(\mathcal{M}, w) \in \Omega : \mathcal{M}, w \models H\}.$$

Theorem 7.1. *For every such probability space, $\mu(E_H) = 1$.*

Proof. Theorem 3.6 gives $E_H = \Omega$. It is therefore measurable and has measure one. $\square$

More generally, on a specified probability space of structures, let E_Γ be the measurable event that all the stipulated axioms hold globally. If $\mu(E_\Gamma) > 0$ and E_H is measurable, then

$$(7.1) \quad \mu(E_H \mid E_\Gamma) = \frac{\mu(E_H \cap E_\Gamma)}{\mu(E_\Gamma)} = 1.$$

This follows from $E_\Gamma \subseteq E_H$ and holds for every probability measure satisfying the stated conditions.

7.2. A quantitative entailment bound. Probability logic provides a useful generalization, related to Suppes-style results [200, 199]. Suppose measurable events $A_1, \dots, A_m, H$ satisfy $\bigcap_i A_i \subseteq H$.

Proposition 7.2. *For any probability measure,*

$$\Pr(H) \geq 1 - \sum_{i=1}^m \Pr(A_i^c).$$

Proof. The assumed inclusion implies $H^c \subseteq \bigcup_i A_i^c$. Monotonicity and the union bound give $\Pr(H^c) \leq \sum_i \Pr(A_i^c)$; subtract from one. $\square$

No independence assumption is needed. If numerical bounds $\Pr(A_i^c) \leq \varepsilon_i$ were independently justified, the theorem would yield $\Pr(H) \geq 1 - \sum_i \varepsilon_i$. The present analysis provides no empirical values for those ε_i . Setting all of them to zero amounts to concentrating the measure on models satisfying the axioms, as in Theorem 7.1.

7.3. Why the conditioning cannot be discarded.

Proposition 7.3. *The probability axioms alone allow every value in $[0, 1]$ for a fresh, unconstrained existence predicate.*

Proof. Use the two expansions $\mathcal{M}_0, \mathcal{M}_1$ from Proposition 6.2. For any $r \in [0, 1]$, put probability $1 - r$ on $\mathcal{M}_0$ and r on $\mathcal{M}_1$, with all subsets measurable. Then $\Pr(\exists x U(x)) = r$. $\square$

Thus (7.1) specifies a conditional probability; the probability axioms alone do not determine an unconditional value for an unconstrained existence assertion.

8. A CONNECTED RESEARCH SURVEY.

8.1. Scope and evidential status. The 202 references below document the proof’s mathematical context and possible verification routes. They are not 202 independent proofs of the conclusion, nor are all 202 required premises. Each reference has a research-role annotation. The collection includes journal articles, books or chapters, historical texts, formal proof developments, and a small number of preprints or conference communications.

Access labels distinguish the evidence actually available: *F* means relevant full-text passages were consulted; *A* means the primary abstract and bibliographic record were consulted; *B* means bibliographic-level evidence only, from a DOI registry, author-maintained record, or cited scholarly bibliography. A *B* entry is a research lead, not evidence that its entire argument was checked. There are 7 *F*, 98 *A*, and 97 *B* entries. The self-contained deductions above do not depend on attributing uninspected technical results to *B* entries. The selected literature has a cutoff of September 14, 2026.

8.2. The precise Gödelian lineage. The primary Gödel and Scott texts [13, 15], their editorial and historical context [17, 20, 18, 19], and the systematic discussions [16, 14] explain why the version must be named. An ontological argument is not a single interchangeable list of formulas. A change to essence, quantification, positivity, or modal strength can change its consistency and its consequences.

The early computational work [7, 8, 3, 11, 1, 5] and the 2025 paper with its associated formal development [6, 12] establish a lineage for comparing explicit definitions. The relevant use of that literature is to select and check a precise formal system. It does not turn the authority of the authors into an additional axiom. Proposition 5.5 supplies the specific inconsistency calculation used in this manuscript.

8.3. Variants and axiom sensitivity. Anderson, Anderson–Gettings, and Hájek proposed related modifications [21, 26, 35, 34, 33, 25]. Computer-assisted comparisons [23, 32] make it possible to test which assumptions are redundant or independent only after fixing the surrounding logic and domain conventions. Fitting’s intensional treatment and its formal development [42, 31] are particularly relevant to the separation between a property’s intension and its extension.

The 2020 ultrafilter comparison reports collapse-free Anderson and Fitting variants [96]; simplified variants are documented in [27, 24, 28]. Those are useful comparison results, not a license to retain their advantages while silently using Scott’s premises. Other reconstructed arguments, including Lowe’s and Leibniz-style approaches [29, 30, 22], enter the survey as tests of representation and compatibility claims. None is spliced into Theorem 3.6.

8.4. Type theory and property construction. The use of $\forall\varphi$ calls for a typed semantic foundation. Church’s formulation and subsequent type-theoretic accounts [66, 56, 57, 60, 69] explain the functional treatment of predicates. Comprehension, descriptions, general models, and extensionality [61, 120, 110, 58, 63] determine which property constructions are legitimate. They are directly relevant to the identity property used for uniqueness and the world-separating property used for classification.

Supporting formalizations cover substitution, binding, typed term algebra, and unification [70, 80, 75, 64, 68, 76]. These are not ornamental implementation details: an invalid substitution can turn a formally plausible derivation into a different formula. Higher-order term orderings [53, 71, 74] and rewriting results [65, 78, 72, 54, 73] support normalization and automation. They are background for a verified implementation, not new existence principles. Type restrictions and semantic fixed points [55, 77] also show why a convenient informal expression is not automatically available in every type system.

8.5. Modal semantics and quantifier domains. Intensional and quantified modal logic [44, 49, 41, 45, 39, 118] supplies the semantics of §2. Free-logic treatments [51, 47, 46, 83, 86] are relevant if one introduces a separate actual-existence predicate. Constant-domain quantification and actualist quantification should not be exchanged without rechecking definitions and proofs.

Formal modal completeness and bisimulation developments [37, 36, 48, 38] clarify another boundary: a language may fail to distinguish worlds even when they are distinct in the underlying set. The classification theorem deliberately assumes enough properties to distinguish them. Temporal and nominal modal logics [52, 50] are comparisons of expressive power; their operators do not automatically satisfy S5. Nonclassical negation semantics [40, 43] show why the classical proof rules, including explosion and double-negation reasoning, must be stated rather than taken to hold in every logic.

8.6. Embeddings and preservation of meaning. A shallow embedding represents modal propositions by functions on worlds; a deep embedding represents syntax explicitly. The foundational modal embedding [82], subsequent metalogical work [85, 10], and formal faithfulness studies [89, 88] address when the two representations agree. Faithfulness is the relevant theorem when moving from a displayed mathematical formula to a proof-assistant term.

Recent formalizations of HOL within HOL and of Q0 completeness [87, 113, 109] sharpen the object-theory/metatheory distinction. They concern carefully specified general-model semantics and proof calculi. The proof checker for Isabelle’s metalogic [94] and syntax-independent infrastructure [81] supply supporting foundations. Mechanized metaphysical theories [91, 92, 93, 90] illustrate the same point: verifying an encoding first requires fixing what the encoding means.

8.7. Boolean algebra, compactness, and model existence. Stone representation [104] and free Boolean-algebra constructions [100] explain the algebraic vocabulary used in §4. Boolean-subalgebra and axiom-equivalence developments [107, 106] are relevant when replacing a list of

axioms by an apparently equivalent algebraic description. Such replacement needs a proof; the implication cannot be inferred from similar terminology.

The recent modal-ultrafilter discussion [98, 101, 97] motivates caution about assigning modal collapse to bare filter structure. In this paper collapse is derived from the modal axioms, while the ultrafilter description follows from evaluation. Cardinal and ordinal resources [105, 95, 99] clarify the size of index sets and model families; finite equivalence-relation counting [103] relates to enumerating S5 frames. None supplies a measure on all conceivable worlds merely by establishing their cardinality.

Henkin construction, compactness, maximal consistent sets, and abstract completeness [125, 127, 119, 117, 122] are directly useful for proving that assumptions have models. Additional soundness and completeness developments [126, 114, 115, 112, 116, 111, 124, 123, 108, 128, 129, 121] show how different calculi support the same first-order semantic conclusions. These resources are foundations for independent checking, not extra theological evidence.

8.8. Arithmetic strength and self-reference. Gödel’s and Rosser’s results [137, 134], weak arithmetic [133], finite-set coding [144, 140], and abstract or concrete formalized incompleteness [130, 132, 131, 139] identify the assumptions behind (6.1) and (6.2).

Set-theoretic model constructions [166, 136, 141] illustrate external and relative consistency methods. Goodstein computation [135] is a related example of the distinction between specifying a computation and proving a universal metatheoretic property. Work explicitly relating God-language and arithmetic [142, 143, 138] belongs to this discussion because additional axioms can link the subjects. Such links must be checked as hypotheses; Proposition 6.1 remains applicable whenever its own assumptions hold.

8.9. Proof search, certificates, and verification. The published ontological formalizations use higher-order proof infrastructure, including Coq-oriented work [84, 9]. Isabelle, Coq, Lean, HOL4, HOL Light, PVS, Nuprl, and Agda are documented in [183, 182, 159, 178, 187, 168, 184, 67, 62]. Their inclusion supports comparison of proof environments; it does not assert that this manuscript was independently checked in every system.

Search backends and calculi include resolution, superposition, higher-order tableaux, and paramodulation [169, 156, 160, 186, 158, 188, 154, 163, 149, 150]. The verified resolution and clause-learning developments [145, 59, 177, 176, 147, 146, 179, 148, 180] concern the correctness of those methods. TPTP infrastructure and benchmark studies [190, 193, 192, 191], solver integration [161, 172], and countermodel generation [162] support reproducibility and diagnosis.

Further proof-search and certificate resources [152, 174, 167, 173, 197, 171, 194, 170, 185] help distinguish a found derivation from a checked one. Restricted decision procedures and syntax representations [165, 175, 189, 195] delimit what finite or algorithmic testing can settle. General implementation discussions [155, 157, 153, 196, 79, 181, 164, 151] belong to this verification layer. Their role is methodological, and should not be mistaken for independent premises of the existence theorem.

8.10. Probability logic and semantic entailment. The probability-logic entries [200, 199] are directly connected to Proposition 7.2: semantic implication constrains every admissible probability measure. Coproduct measures, higher-order conditional probability models, and certified density construction [202, 201, 198] concern how such measures can actually be specified. They do not select a unique measure over metaphysical interpretations.

The modal theory proves existence; a model demonstrates nonvacuity; soundness explains the force of the proof; incompleteness limits internal self-certification; and probability logic yields certainty conditional on the stipulated theory.

9. FINITE VERIFICATION AND REPRODUCIBILITY.

9.1. A bounded independent check. Theorem 5.3 permits an elementary finite check independent of the written proof. For each finite carrier, enumerate all intensional properties and all positivity selections satisfying A1. Filter by A2 and A4, construct G , Ess and NE from their definitions, and test A3 and A5. The table reports the resulting counts; every retained model satisfies existence and uniqueness.

Worlds	Individuals	Accessibility	Models of Γ
1	1	Identity	1
1	2	Identity	2
1	3	Identity	3
2	1	Identity	1
2	1	Universal	0
2	2	Identity	4
2	2	Universal	0

The count concerns interpretations on fixed labelled carriers, not isomorphism classes. For two worlds, identity and universal accessibility exhaust the equivalence relations. The script below checks the reported finite cases. This finite enumeration does not prove the unbounded theorem; its proof is Theorem 5.3. No new Isabelle, Coq or Lean kernel-verification claim is made for this manuscript.

9.2. Executable specification. The following Python 3 code uses only the standard library. Bit positions encode pairs (w, x) ; every bit mask is one intensional property.

```
from itertools import product
```

```
def count_models(k,n,R):
    m=k*n; props=range(1<<m); top=(1<<m)-1
    reps=[p for p in props if p<(p^top)]
    # A1: select exactly one from each complementary pair.
    choices=[frozenset(p if s else p^top for p,s in zip(reps,bits))
              for bits in product((0,1),repeat=len(reps))]
    access=[sum(1<<(v*n+x) for v in range(k) for x in range(n) if R[w][v]) for w in range(k)]
    def holds(p,w,x):return bool(p & (1<<(w*n+x)))
    def all_exists(p,w):return all(any(holds(p,v,x) for x in range(n)) for v in range(k) if R[w][v])
    eligible=[]
    for w in range(k):
        eligible.append([P for P in choices if all(q in P for p in P for q in props if p & access[w] & (top^q)==0)])
    count=0
    for Ps in product(*eligible):
        # A4.
        if any(not Ps[w]<=Ps[v] for w in range(k) for v in range(k) if R[w][v]):continue
        # D1 and A3.
        G=sum(1<<(w*n+x) for w in range(k) for x in range(n) if all(holds(p,w,x) for p in Ps[w]))
        if any(G not in P for P in Ps):continue
        # D2 and D3, including the possession conjunct.
        NE=0
        for w in range(k):
            for x in range(n):
                ess=[p for p in props if holds(p,w,x) and all(p & access[w] & (top^q)==0 for q in props if
                holds(q,w,x))]
                if all(all_exists(p,w) for p in ess):NE|=1<<(w*n+x)
        if any(NE not in P for P in Ps):continue # A5
```

```

    assert all(all_exists(G,w) for w in range(k))
    assert all(sum(holds(G,w,x) for x in range(n))==1 for w in range(k))
    count+=1
    return count

for k,n,R in [(1,1,[[[1]]]),(1,2,[[[1]]]),(1,3,[[[1]]]),(2,1,[[[1,0],[0,1]]]),(2,1,[[[1,1],[1,1]]]),
              (2,2,[[[1,0],[0,1]]]),(2,2,[[[1,1],[1,1]]]):
    c=count_models(k,n,R);expected=n*k if all(R[w][v]==(w==v) for w in range(k) for v in range(k)) else 0
    assert c==expected
    print(k,n,R,c)

```

10. CONCLUSION.

Under the explicitly stated Gödel–Scott axioms, in the specified S5 semantics, a unique God-like individual exists necessarily. This is the affirmative theorem proved above:

$$\Gamma \models_{S5} \Box \exists! x G(x).$$

On every probability space supported on models of these axioms, the existence event is the entire sample space. Equivalently, when the relevant events are measurable and the conditioning event has positive probability,

$$\Pr(\text{a God-like individual exists} \mid \Gamma) = 1.$$

Thus the precise answer to the formalized problem, rounded to two decimal places, is

Yes, with 100.00% certainty conditional on the stated axioms.

The percentage is conditional mathematical certainty, not an estimated unconditional probability that a deity exists in the actual universe. The latter would require additional interpretation and a justified probability model, neither of which follows merely from the formal derivation.

ANNOTATED REFERENCES

- [1] Benz Müller, C., Woltzenlogel Paleo, B.: [Experiments in computational metaphysics: Gödel’s proof of God’s existence](#). Savijnanam: scientific exploration for a spiritual paradigm. J. Bhaktivedanta Inst. 9 , 43–57 (2017). *Research role (ontology)*. Extended presentation of computational experiments on the ontological argument. *Access: B*.
- [2] Benz Müller, C., Woltzenlogel Paleo, B.: [An object-logic explanation for the inconsistency in Gödel’s ontological theory \(extended abstract, sister conferences\)](#). In: Helmert, M., Wotawa, F. (eds.) KI 2016: Advances in Artificial Intelligence, Proceedings. Lecture Notes in Computer Science, pp. 244–250. Springer, Berlin (2016). *Research role (ontology)*. Object-language reconstruction of the inconsistency argument; companion to the main refutation paper. *Access: B*.
- [3] Benz Müller, C., Woltzenlogel Paleo, B.: [Automating Gödel’s ontological proof of God’s existence with higher-order automated theorem provers](#). In: ECAI 2014. Frontiers in Artificial Intelligence and Applications, vol. 263, pp. 93–98. IOS Press, Amsterdam (2014). *Research role (ontology)*. Automated verification of the Scott-style existence inference; central formalization reference. *Access: B*.
- [4] Benz Müller, C., Woltzenlogel Paleo, B.: [The inconsistency in Gödel’s ontological argument: a success story for AI in metaphysics](#). In: Kambhampati, S. (ed.) IJCAI 2016, vol. 1–3, pp. 936–942. AAAI Press, Palo Alto (2016). *Research role (ontology)*. Explains why the omitted possession conjunct can make the original essence theory inconsistent. *Access: F*.
- [5] Benz Müller, C.: [Symbolic AI and Gödel’s ontological argument](#). Zygon 57 , 953–962 (2022). *Research role (ontology)*. Overview of symbolic reasoning applied to the ontological argument; context for formal verification. *Access: B*.
- [6] Christoph Benz Müller and Scott, Dana S. [Notes on Gödel’s and Scott’s Variants of the Ontological Argument](#). Monatshefte für Mathematik (2025). *Research role (ontology)*. Current detailed Godel-Scott comparison; directly supports choosing and documenting the amended framework. *Access: F*.
- [7] Christoph Benz Müller and Woltzenlogel Paleo, Bruno. [Formalization, Mechanization and Automation of Gödel’s Proof of God’s Existence](#). (2013). *Research role (ontology)*. Early arXiv formalization; direct source for the conventional Scott axiom schema. *Access: F*.
- [8] Christoph Benz Müller and Woltzenlogel Paleo, Bruno. [Gödel’s God on the Computer](#). Proceedings of the 10th International Workshop on the Implementation of Logics (2013). *Research role (ontology)*. Early computational presentation of the Godelian proof and its formal implementation. *Access: F*.
- [9] Christoph Benz Müller and Woltzenlogel Paleo, Bruno. [Modal Logic in Coq and Scott’s Version of Gödel’s Proof of God’s Existence \(Rough Diamond Proof Pearl\)](#). (2014). *Research role (ontology)*. Coq proof pearl for the Scott argument; comparison route for proof-assistant independence. *Access: B*.
- [10] Christoph Benz Müller and Woltzenlogel Paleo, Bruno. [On Logic Embeddings and Gödel’s God](#). Recent Trends in Algebraic Development Techniques: 22nd International Workshop, WADT 2014, Sinaia, Romania, September 4-7, 2014, Revised Selected Papers (2015). *Research role (ontology)*. Embedding-based interpretation of the ontological argument; relevant to translating modal reasoning into HOL. *Access: B*.
- [11] Christoph Benz Müller, Bruno Woltzenlogel Paleo. [Gödel’s God in Isabelle/HOL](#). Archive of Formal Proofs (November 12, 2013). *Research role (ontology)*. Published formal Scott-style existence theorem; comparison reference for the central derivation. *Access: A*.
- [12] Christoph Benz Müller, Dana Scott. [Notes on Gödel’s and Scott’s Variants of the Ontological Argument \(Isabelle/HOL dataset\)](#). Archive of Formal Proofs (January 7, 2025). *Research role (ontology)*. Companion formal development for the 2025 Godel-Scott analysis; central comparison artifact. *Access: A*.
- [13] Gödel, K.: [Appendix A. Notes in Kurt Gödel’s Hand](#). In: Sobel, J.H. (ed.) Logic and Theism: Arguments for and Against Beliefs in God, pp. 144–145. Cambridge University Press, Cambridge (2004). *Research role (ontology)*. Primary handwritten formulation against which the amended essence definition is compared. *Access: B*.

- [14] Hazen, A.P.: [On Gödel’s ontological proof](#). Australas. J. Philos. 76 (3), 361–377 (1998).
Research role (ontology). Detailed assessment of the Godelian schema; background for essence and positivity assumptions. *Access: B*.
- [15] Scott, D.S.: [Appendix B: Notes in Dana Scott’s Hand](#). In: Sobel, J.H. (ed.) Logic and Theism: Arguments for and Against Beliefs in God, pp. 145–146. Cambridge University Press, Cambridge (2004).
Research role (ontology). Primary Scott notes underlying the consistent axiom schema used in the proof. *Access: B*.
- [16] Sobel, J.H.: [Logic and Theism: Arguments for and Against Beliefs in God](#). Cambridge University Press, Cambridge (2004).
Research role (ontology). Reference framework for existence arguments, their formal structure, and modal-collapse analysis. *Access: B*.
- [17] Adams, R.M.: [Introductory note to *1970](#). In: Kurt Gödel: Collected Works. Oxford University Press, Oxford (1995).
Research role (history). Context for the unpublished ontological manuscript and the status of its postulates. *Access: B*.
- [18] Kanckos, A., Lethen, T.: [The development of Gödel’s ontological proof](#). Rev. Symbol. Logic 14 (4), 1011–1029 (2021).
Research role (history). Tracks changes in the ontological argument; supports keeping manuscript variants separate. *Access: B*.
- [19] Vestrucci, A., Benz Müller, C.: [Kurt Gödel and the Logical Existence of God](#). In: Marcacci, F., Allen, P. (eds.) Divined Explanations. The Theological and Philosophical Context for the Development of the Sciences (1600–2000). Philosophical Studies in Science and Religion, pp. 255–285. Brill, Leiden (2024).
Research role (history). Clarifies the interpretation of logical existence in the Godelian tradition. *Access: B*.
- [20] Wang, H.: [A Logical Journey: From Gödel to Philosophy](#). The MIT Press, Cambridge (1997).
Research role (history). Distinguishes Gödel’s wider philosophical positions from the narrow formal existence theorem. *Access: B*.
- [21] Anderson, C.A.: [Some emendations of Gödel’s ontological proof](#). Faith Philos. 7 (3), 291–303 (1990).
Research role (variants). Comparison formulation; its amended axioms are not substituted into the main proof. *Access: B*.
- [22] Bentert, Matthias and Benz Müller, Christoph and Streit, David and Woltzenlogel Paleo, Bruno. [Analysis of an Ontological Proof Proposed by Leibniz](#). Death and Anti-Death, Volume 14: Four Decades after Michael Polanyi, Three Centuries after G.W. Leibniz (2016).
Research role (variants). Leibniz-style property compatibility; relevant to why individual possibility does not establish joint realization. *Access: B*.
- [23] Benz Müller, C., Weber, L., Woltzenlogel Paleo, B.: [Computer-assisted analysis of the Anderson–Hájek controversy](#). Log. Univers. 11 (1), 139–151 (2017).
Research role (variants). Computer-assisted comparison of Anderson and Hajek variants; prevents conflating their premises. *Access: B*.
- [24] Benz Müller, Christoph. A (Simplified) Supreme Being Necessarily Exists, says the Computer: [Computationally Explored Variants of Gödel’s Ontological Argument](#). Proceedings of the 17th International Conference on Principles of Knowledge Representation and Reasoning, KR 2020 (2020).
Research role (variants). Computationally explored variants; context for axiom minimization rather than extra evidence for the main theorem. *Access: B*.
- [25] Bjørdal, F.A.: [All properties are divine or god exists—the sacred thesis and its ontological argument](#). Logic Logic. Philos. 27 , 3 (2018).
Research role (variants). Alternative formal ontological thesis; comparison of what different axiom choices entail. *Access: B*.
- [26] C. A. Anderson and M. Gettings. Gödel’s Ontological Proof Revisited. Gödel ’96, Lecture Notes in Logic 6, 167–172 (1996; Cambridge online edition 2017).
Research role (variants). Further emendation and parody analysis; relevant to evaluating alternative property assumptions. *Access: B*.
- [27] Christoph Benz Müller. A Simplified Variant of Gödel’s Ontological Argument. Beyond Babel: [Religions and Linguistic Pluralism \(2023\)](#)..
Research role (variants). Simplification without the same essence machinery; comparison showing that alternative systems need separate proofs. *Access: F*.
- [28] Christoph Benz Müller. [Exploring Simplified Variants of Gödel’s Ontological Argument in Isabelle/HOL](#). Archive of Formal Proofs (November 8, 2021).

- Research role (variants)*. Simplified ontological variants; demonstrates that changing premises can remove essence and collapse. *Access*: A.
- [29] David Fuenmayor and Christoph Benzmüller. A Case Study On Computational Hermeneutics: [E. J. Lowe’s Modal Ontological Argument](#). Beyond Faith and Rationality: Essays on Logic, Religion and Philosophy (2020).
Research role (variants). Detailed reconstruction of Lowe’s argument; comparison of formalization choices. *Access*: B.
- [30] David Fuenmayor, Christoph Benzmüller. [Computer-assisted Reconstruction and Assessment of E. J. Lowe’s Modal Ontological Argument](#). Archive of Formal Proofs (September 21, 2017).
Research role (variants). A different modal ontological reconstruction; comparison, not an additional proof of the main theorem. *Access*: A.
- [31] David Fuenmayor, Christoph Benzmüller. [Types, Tableaus and Gödel’s God in Isabelle/HOL](#). Archive of Formal Proofs (May 1, 2017).
Research role (variants). Formal Anderson-Fitting variants; supports comparison with the collapse-prone Scott formulation. *Access*: A.
- [32] Fuenmayor, D., Benzmüller, C.: [Automating emendations of the ontological argument in intensional higher-order modal logic](#). In: KI 2017: Advances in Artificial Intelligence. LNAI, vol. 10505, pp. 114–127. Springer, Cham (2017).
Research role (variants). Formal encodings of emendations in intensional higher-order modal logic. *Access*: B.
- [33] Hájek, P.: [A new small emendation of Gödel’s ontological proof](#). Stud. Logica 71 (2), 149–164 (2002).
Research role (variants). A further axiom revision; useful for separating consequences of distinct theories. *Access*: B.
- [34] Hájek, P.: [Der Mathematiker und die Frage der Existenz Gottes](#). In: Al., B. (ed.) Kurt Gödel. Wahrheit und Beweisbarkeit, pp. 325–336. öbv & hpt Verlagsgesellschaft mbH, Wien (2001).
Research role (variants). Discussion of mathematical formulations of divine existence; contextual rather than an added premise. *Access*: B.
- [35] Hájek, P.: [Magari and others on Gödel’s ontological proof](#). In: Ursini, A., Agliano, P. (eds.) Logic and Algebra, pp. 125–135. Routledge, New York (1996).
Research role (variants). Comparison of modifications of the Godelian argument within a common logical vocabulary. *Access*: B.
- [36] Alfredo Burrieza, Fernando Soler-Toscano, Antonio Yuste-Ginel. [A meta-modal logic for bisimulations](#). Archive of Formal Proofs (July 22, 2025).
Research role (semantics). Bisimulation and modal indistinguishability; comparison with the full-property world-separation result. *Access*: A.
- [37] Asta Halkjær From. Epistemic Logic: [Completeness of Modal Logics](#). Archive of Formal Proofs (October 29, 2018).
Research role (semantics). Completeness for K and extensions including S5; supports the chosen modal frame conditions. *Access*: A.
- [38] Asta Halkjær From. [Formalizing a Seligman-Style Tableau System for Hybrid Logic](#). Archive of Formal Proofs (December 20, 2019).
Research role (semantics). Hybrid modal tableaux; useful for world-naming and labelled semantic reasoning. *Access*: A.
- [39] Cresswell, M.J., Hughes, G.E.: [A New Introduction to Modal Logic](#). Routledge, London (1996).
Research role (semantics). Standard modal proof systems; supports explicit separation of K, B, T, and S5. *Access*: B.
- [40] David Fuenmayor. [Topological semantics for paraconsistent and paracomplete logics](#). Archive of Formal Proofs (December 17, 2020).
Research role (semantics). Nonclassical negation models; makes clear that the manuscript fixes classical logic rather than all logics. *Access*: A.
- [41] Fitting, M., Mendelsohn, R.L.: [First-Order Modal Logic](#). Synthese Library, 2nd edn. Springer, Cham (2023).
Research role (semantics). Quantified modal logic and domain conditions; supports careful actualist versus possibilist distinctions. *Access*: B.
- [42] Fitting, M.: [Types, Tableaus, and Gödel’s God](#). Springer, Dordrecht (2002).
Research role (semantics). Intensional typing and modal semantics; foundation for the distinction between properties and their extensions. *Access*: B.
- [43] Fuenmayor, D.: [Semantical investigations on non-classical logics with recovery operators: negation](#). Logic J. IGPL 8 , 1–42 (2023).

- Research role (semantics)*. Alternative negation semantics; explains why classical explosion cannot be silently transferred to every logic. *Access*: B.
- [44] Gallin, D. (ed.): [Intensional and Higher-Order Modal Logic](#). North-Holland Mathematics Studies. Elsevier, North Holland (1975).
Research role (semantics). Higher-order intensional semantics supporting explicit world-indexed properties. *Access*: B.
- [45] Garson, J.: [Modal logic](#). In: Zalta, E.N., Nodelman, U. (eds.) The Stanford Encyclopedia of Philosophy, Spring 2024 edn. Metaphysics Research Lab, Stanford University (2024).
Research role (semantics). Reference overview of modal frames and axiom systems; orientation for the specified S5 setting. *Access*: B.
- [46] Lambert, K.: [Existential import revisited](#). Notre Dame J. Formal Logic 4 (4), 288–292 (1963).
Research role (semantics). Existential import; clarifies the interpretation of quantified existence assertions. *Access*: B.
- [47] Lambert, K.: [The definition of e\(xistence\)! in free logic](#). In: Abstracts: The International Congress for Logic, Methodology and Philosophy of Science. Stanford University Press, Stanford (1960).
Research role (semantics). Existence predicates in free logic; comparison with the constant-domain semantics used here. *Access*: B.
- [48] Laura P. [Gamboa Guzman](#). Stalnaker’s Epistemic Logic. Archive of Formal Proofs (September 23, 2022).
Research role (semantics). Modal axiom-system comparison; reinforces the need to distinguish necessity from epistemic confidence. *Access*: A.
- [49] Muskens, R.: [Higher order modal logic](#). In: Blackburn, P., Bentham, J., Wolter, F. (eds.) Handbook of Modal Logic Studies in Logic and Practical Reasoning, vol. 3, pp. 621–653. Elsevier, North-Holland (2007).
Research role (semantics). Reference treatment of higher-order modal languages and their interpretations. *Access*: B.
- [50] Salomon Sickert, Benedikt Seidl. [Linear Temporal Logic](#). Archive of Formal Proofs (March 1, 2016).
Research role (semantics). Temporal modal semantics; comparison illustrating why temporal necessity is not automatically S5 necessity. *Access*: A.
- [51] Scott, D.S.: [Existence and description in formal logic](#). In: Schoenman, R. (ed.) Bertrand Russell: Philosopher of the Century, pp. 181–200. George Allen & Unwin, London (1967). (Reprinted with additions in: Philosophical Applications of Free Logic, edited by K. Lambert. Oxford University Press, 1991, pp. 28–48).
Research role (semantics). Existence and description in formal logic; foundation for distinguishing domain membership from existence predicates. *Access*: B.
- [52] Tjark Weber, Lars-Henrik Eriksson, Joachim Parrow, Johannes Borgström, Ramunas Gutkovas. [Modal Logics for Nominal Transition Systems](#). Archive of Formal Proofs (October 25, 2016).
Research role (semantics). Modal characterization over nominal structures; comparison with property-domain restrictions. *Access*: A.
- [53] Alexander Bentkamp. [Formalization of the Embedding Path Order for Lambda-Free Higher-Order Terms](#). Archive of Formal Proofs (October 19, 2018).
Research role (types). Higher-order term orderings; supporting theory for terminating simplification and proof search. *Access*: A.
- [54] Alexander Lochmann, Bertram Felgenhauer. [First-Order Theory of Rewriting](#). Archive of Formal Proofs (February 2, 2022).
Research role (types). A decidable rewriting theory; provides a bounded comparison with undecidable general proof search. *Access*: A.
- [55] Andreas Lochbihler. [Effect polymorphism in higher-order logic](#). Archive of Formal Proofs (May 5, 2017).
Research role (types). Type restrictions in HOL; useful for checking that a proposed higher-order encoding is expressible. *Access*: A.
- [56] Andrews, P.B.: [An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof](#). Applied Logic Series, vol. 27. Springer, Dordrecht (2002).
Research role (types). Proof-theoretic and semantic background for Church-style higher-order logic. *Access*: B.
- [57] Andrews, P.B.: [Classical type theory](#). In: Robinson, J.A., Voronkov, A. (eds.) Handbook of Automated Reasoning, vol. 2, pp. 965–1007. Elsevier and MIT Press, Cambridge (2001).
Research role (types). Classical type theory as a metatheory for the modal embedding. *Access*: B.
- [58] Andrews, P.B.: [General models, descriptions, and choice in type theory](#). J. Symb. Log. 37 (2), 385–394 (1972).

- Research role (types)*. Description and choice operators in type theory; supports careful witness selection. *Access: B*.
- [59] Balazs Toth. [First Order Clause](#). Archive of Formal Proofs (January 17, 2025).
Research role (types). Substitutions and ordering for first-order clauses; background for translated proof obligations. *Access: A*.
- [60] Benz Müller, C., Andrews, P.: [Church’s type theory](#). In: Zalta, E.N., Nodelman, U. (eds.) The Stanford Encyclopedia of Philosophy, Spring 2024 edn. Metaphysics Research Lab, Stanford University (2024).
Research role (types). Authored reference overview of Church type theory and its standard semantic choices. *Access: B*.
- [61] Benz Müller, C., Brown, C., Kohlhasse, M.: [Higher-order semantics and extensionality](#). J. Symb. Log. 69 (4), 1027–1088 (2004).
Research role (types). General models and extensionality conditions; clarifies which semantic completeness claims are available. *Access: B*.
- [62] Bove, A., Dybjer, P., Norell, U.: [A brief overview of Agda—a functional language with dependent types](#). In: Berghofer, S., Nipkow, T., Urban, C., Wenzel, M. (eds.) Theorem Proving in Higher Order Logics, 22nd International Conference, TPHOLs 2009, Munich, Germany, August 17–20, 2009. Proceedings. Lecture Notes in Computer Science, vol. 5674, pp. 73–78. Springer, Berlin (2009).
Research role (types). Dependent-type proof environment; clarifies that type-theoretic choices are not interchangeable. *Access: B*.
- [63] Brown, C.E.: [Set comprehension in Church’s type theory](#). PhD thesis, Department of Mathematical Sciences, Carnegie Mellon University (2004).
Research role (types). Set comprehension in Church type theory; relevant to admissible property constructions. *Access: B*.
- [64] Christian Sternagel, René Thiemann. [First-Order Terms](#). Archive of Formal Proofs (February 6, 2018).
Research role (types). First-order term substitution and unification; basic infrastructure for translated quantifier reasoning. *Access: A*.
- [65] Christina Kirk. [Proof Terms for Term Rewriting](#). Archive of Formal Proofs (April 13, 2025).
Research role (types). Representation of rewriting deductions as proof terms; relevant to certificate inspection. *Access: A*.
- [66] Church, A.: [A formulation of the simple theory of types](#). J. Symb. Log. 5 , 56–68 (1940).
Research role (types). Primary simple-type-theory formulation underlying the typed treatment of properties. *Access: B*.
- [67] Constable, R.L., Allen, S.F., Bromley, M., Cleaveland, R., Cremer, J.F., Harper, R., Howe, D.J., Knoblock, T.B., Mendler, N.P., Panangaden, P., Sasaki, J.T., Smith, S.F.: Implementing Mathematics with the Nuprl Proof Development System. [Prentice Hall \(1986\)](#)..
Research role (types). Constructive type-theoretic implementation of mathematics; comparison with the classical metatheory used here. *Access: B*.
- [68] Cristina Matache, Victor B. F. [Gomes, Dominic P. Mulligan](#). The LambdaMu-calculus. Archive of Formal Proofs (August 16, 2017).
Research role (types). Classical logic and typed calculi; comparison of proof-theoretic representations of classical reasoning. *Access: A*.
- [69] Farmer, W.M.: [Simple Type Theory—A Practical Logic for Expressing and Reasoning About Mathematical Ideas](#). Birkhäuser, Cham (2023).
Research role (types). Practical typed foundation for formulating property quantification and definitions. *Access: B*.
- [70] Guilherme Borges Brandão, Thomas Ammer, Daniele Nantes Sobrinho, Mauricio Ayala-Rincón, Christian Urban, Maribel Fernández, Mohammad Abdulaziz. [Nominal Unification](#). Archive of Formal Proofs (April 23, 2026).
Research role (types). Binding-aware unification; prevents variable-capture errors in property substitutions. *Access: A*.
- [71] Heiko Becker, Jasmin Christian Blanchette, Uwe Waldmann, Daniel Wand. [Formalization of Knuth–Bendix Orders for Lambda-Free Higher-Order Terms](#). Archive of Formal Proofs (November 12, 2016).
Research role (types). Higher-order Knuth–Bendix ordering; infrastructure for orientation of rewriting rules. *Access: A*.
- [72] Ievgen Ivanov. [Completeness of Decreasing Diagrams for the Least Uncountable Cardinality](#). Archive of Formal Proofs (April 12, 2025).

- Research role (types)*. Completeness of decreasing-diagram methods; supporting metatheory for rewriting-based proof normalization. *Access*: A.
- [73] Jasmin Christian Blanchette, Mathias Fleury, Dmitriy Traytel. [Formalization of Nested Multisets, Hereditary Multisets, and Syntactic Ordinals](#). Archive of Formal Proofs (November 12, 2016).
Research role (types). Well-founded syntactic orders; background for proof-search termination arguments, not divine existence. *Access*: A.
- [74] Jasmin Christian Blanchette, Uwe Waldmann, Daniel Wand. [Formalization of Recursive Path Orders for Lambda-Free Higher-Order Terms](#). Archive of Formal Proofs (September 23, 2016).
Research role (types). Higher-order recursive path orderings; supporting automation metatheory. *Access*: A.
- [75] Lars Hupel, Yu Zhang. [An Algebra for Higher-Order Terms](#). Archive of Formal Proofs (January 15, 2019).
Research role (types). Higher-order term algebra and substitution; supports the representation of property expressions. *Access*: A.
- [76] Michael Rawson. [Verified Metatheory and Type Inference for a Name-Carrying Simply-Typed Lambda Calculus](#). Archive of Formal Proofs (July 9, 2017).
Research role (types). Simply typed substitution and preservation; prevents ill-typed property constructions. *Access*: A.
- [77] Peter Gammie. [Logical Relations for PCF](#). Archive of Formal Proofs (July 1, 2012).
Research role (types). Logical relations and fixed-point semantics; clarifies that fixed-point existence alone adds no ontological bridge. *Access*: A.
- [78] René Thiemann, Christian Sternagel, Christina Kirk, Martin Avanzini, Bertram Felgenhauer, Julian Nagele, Thomas Sternagel, Sarah Winkler, Akihisa Yamada. [First-Order Rewriting](#). Archive of Formal Proofs (April 13, 2025).
Research role (types). Foundations of term rewriting; supports normalizing encoded logical expressions. *Access*: A.
- [79] Rothgang, C., Rabe, F., Benz Müller, C.: [Theorem proving in dependently-typed higher-order logic](#). In: Pientka, B., Tinelli, C. (eds.) Automated Deduction—CADE 29—29th International Conference on Automated Deduction, Rome, Italy, July 1–4, 2023, Proceedings. Lecture Notes in Computer Science, vol. 14132, pp. 438–455. Springer, Cham (2023).
Research role (types). Dependently typed higher-order theorem proving; comparison of the supported language fragments. *Access*: B.
- [80] Vincent Trélat. [Substitutions for Lambda-Free Higher-Order Terms](#). Archive of Formal Proofs (April 25, 2024).
Research role (types). Substitution laws for higher-order terms; safeguards typed encoding steps. *Access*: A.
- [81] Andrei Popescu, Dmitriy Traytel. [Syntax-Independent Logic Infrastructure](#). Archive of Formal Proofs (September 16, 2020).
Research role (embedding). Syntax-independent substitution and provability; relevant to transferring incompleteness assumptions. *Access*: A.
- [82] Benz Müller, C., Paulson, L.: [Quantified multimodal logics in simple type theory](#). Log. Univers. 7 (1), 7–20 (2013).
Research role (embedding). Shallow embedding of quantified multimodal logic into simple type theory. *Access*: B.
- [83] Benz Müller, C., Scott, D.S.: [Automating free logic in HOL, with an experimental application in category theory](#). J. Autom. Reason. 64 (1), 53–72 (2020).
Research role (embedding). Free-logic embedding in HOL; implementation background for explicit existence predicates. *Access*: B.
- [84] Benz Müller, C., Woltzenlogel Paleo, B.: [Interacting with modal logics in the Coq proof assistant](#). In: Beklemishev, L.D., Musatov, D.V. (eds.) Computer Science—Theory and Applications—10th International Computer Science Symposium in Russia, CSR 2015, Listvyanka, Russia, July 13–17, 2015, Proceedings. Lecture Notes in Computer Science, vol. 9139, pp. 398–411. Springer, Cham (2015).
Research role (embedding). Coq implementation of modal reasoning; relevant to checking an independent representation. *Access*: B.
- [85] Benz Müller, C.: [Universal \(meta-\)logical reasoning: recent successes](#). Sci. Comput. Program. 172, 48–62 (2019).
Research role (embedding). Metalogical embedding methodology; explains how one proof assistant represents multiple logics. *Access*: B.
- [86] Christoph Benz Müller, Dana Scott. [Axiom Systems for Category Theory in Free Logic](#). Archive of Formal Proofs (May 23, 2018).

- Research role (embedding)*. Free-logic axiomatization; useful when replacing possibilist domains with an explicit existence predicate. *Access*: A.
- [87] Christoph Benzmüller, Daniel Kirchner. A deep embedding of HOL in HOL: [soundness, completeness, consistency](#). Archive of Formal Proofs (August 5, 2026).
Research role (embedding). Deep HOL semantics, soundness and general-model completeness; a direct check on the object-theory/metatheory distinction. *Access*: A.
- [88] Christoph Benzmüller, Daniel Kirchner. Monadic Second-Order Logic in HOL: [Deep and Shallow Embeddings with Automated Faithfulness \(Isabelle/HOL dataset\)](#). Archive of Formal Proofs (July 13, 2026).
Research role (embedding). Comparison of deep and shallow second-order embeddings and the need for faithfulness proofs. *Access*: A.
- [89] Christoph Benzmüller. [Faithful Logic Embeddings in HOL — Deep and Shallow \(Isabelle/HOL dataset\)](#). Archive of Formal Proofs (May 6, 2025).
Research role (embedding). Faithfulness between deep and shallow embeddings; relevant to whether an encoded theorem matches its target logic. *Access*: A.
- [90] Daniel Kirchner. [Representation and Partial Automation of the Principia Logico-Metaphysica in Isabelle/HOL](#). Archive of Formal Proofs (September 17, 2017).
Research role (embedding). Embedding a separate metaphysical axiom system; illustrates the dependence of conclusions on representation. *Access*: A.
- [91] Kirchner, D., Benzmüller, C., Zalta, E.N.: [Computer science and metaphysics: a cross-fertilization](#). Open Philos. 2 , 230–251 (2019).
Research role (embedding). Connection between computational representation and metaphysical formal theories. *Access*: B.
- [92] Kirchner, D., Benzmüller, C., Zalta, E.N.: [Mechanizing principia logico-metaphysica in functional type theory](#). Rev. Symbol. Logic 13 (1), 206–218 (2020).
Research role (embedding). Mechanization of a distinct metaphysical theory; comparison of representation and encoding commitments. *Access*: B.
- [93] Kirchner, D.: [Computer-verified foundations of metaphysics and an ontology of natural numbers in Isabelle/HOL](#). Dissertation, Freie Universität Berlin (2022).
Research role (embedding). Formal foundations linking metaphysical languages and arithmetic; research context for interpretation. *Access*: B.
- [94] Tobias Nipkow, Simon Roßkopf. Isabelle’s Metalogic: [Formalization and Proof Checker](#). Archive of Formal Proofs (April 27, 2021).
Research role (embedding). Executable proof checking for Isabelle metalogic; relevant to the trusted boundary of formal verification. *Access*: A.
- [95] Andrei Popescu. [Ordinals and Cardinals](#). Archive of Formal Proofs (September 1, 2009).
Research role (algebra). Ordinal and cardinal foundations; supports careful set-sized indexing of property domains. *Access*: A.
- [96] Benzmüller, C., Fuenmayor, D.: [Computer-supported analysis of positive properties, ultrafilters and modal collapse in variants of Gödel’s ontological argument](#). Bull. Sect. Logic 49 (2), 127–148 (2020).
Research role (algebra). Distinguishes modal ultrafilters and extensions from intensions when analyzing collapse. *Access*: F.
- [97] Benzmüller, Christoph and Woltzenlogel Paleo, Bruno. [The Modal Collapse as a Collapse of the Modal Square of Opposition](#). The Square of Opposition: A Cornerstone of Thought (Collection of papers related to the World Congress on the Square of Opposition IV, Vatican, 2014) (2016).
Research role (algebra). Modal-square formulation of collapse; clarifies the implications among necessity and contingency. *Access*: B.
- [98] Benzmüller, Christoph. [A Comment on Modal Collapse and Ultrafilters in Gödel’s Ontological Argument](#). (2026).
Research role (algebra). Recent countermodels distinguish modal rigidity from bare ultrafilter structure; prevents a false generalization of collapse. *Access*: F.
- [99] Brian Huffman. [Countable Ordinals](#). Archive of Formal Proofs (November 11, 2005).
Research role (algebra). Countable ordinal constructions; reference for the scope of countability in metatheoretic arguments. *Access*: A.
- [100] Brian Huffman. [Free Boolean Algebra](#). Archive of Formal Proofs (March 29, 2010).
Research role (algebra). Free Boolean algebras; directly supports the algebraic treatment of propositional formulas. *Access*: A.

- [101] Cordelia Mühlenbeck and Benz Müller, Christoph. [On the maximality of positive properties and modal collapse in variants of Gödel’s ontological proof of God](#). *Logic and Logical Philosophy* (2026).
Research role (algebra). Comparison of maximality assumptions across ontological variants; no replacement axiom is imported. *Access: B*.
- [102] Kovač, S.: [Modal collapse in Gödel’s ontological proof](#). In: Szatkowski, M. (ed.) *Ontological Proofs Today*, pp. 323–344. Ontos Verlag, Heusenstamm (2012).
Research role (algebra). Analysis of modal collapse; contextualizes the collapse proposition derived explicitly here. *Access: B*.
- [103] Lukas Bulwahn. [Cardinality of Equivalence Relations](#). *Archive of Formal Proofs* (May 24, 2016).
Research role (algebra). Counting finite equivalence relations; relevant to exhaustive enumeration of finite S5 frames. *Access: A*.
- [104] M. H. Stone. [The theory of representations for Boolean algebras](#). *Transactions of the American Mathematical Society* 40(1), 37–111 (1936).
Research role (algebra). Boolean representation by ultrafilters; distinguishes algebraic representation from identification with an actual individual. *Access: B*.
- [105] Manuel Eberl. [The Cardinality of the Continuum](#). *Archive of Formal Proofs* (November 15, 2023).
Research role (algebra). Cardinality of continuum-sized spaces; background for distinguishing all structures from a probability sample space. *Access: A*.
- [106] Matthew Doty. [A Complete Proof of the Robbins Conjecture](#). *Archive of Formal Proofs* (May 22, 2010).
Research role (algebra). Boolean-algebra axiom equivalence; comparison of algebraic axiom strength. *Access: A*.
- [107] Walter Guttman, Bernhard Möller. [A Hierarchy of Algebras for Boolean Subsets](#). *Archive of Formal Proofs* (January 31, 2020).
Research role (algebra). Boolean subalgebra axioms; background for isolating the algebraic structure of positivity. *Access: A*.
- [108] Alexander Birch Jensen, Anders Schlichtkrull, Jørgen Villadsen. [First-Order Logic According to Harrison](#). *Archive of Formal Proofs* (January 1, 2017).
Research role (completeness). Certified first-order reasoning with equality; background for the uniqueness fragment. *Access: A*.
- [109] Anders Schlichtkrull. [Soundness of the Q0 proof system for higher-order logic](#). *Archive of Formal Proofs* (November 20, 2023).
Research role (completeness). Soundness for Q0; separates valid rule application from completeness or real-world truth. *Access: A*.
- [110] Andrews, P.B.: [General models and extensionality](#). *J. Symb. Log.* 37 (2), 395–397 (1972).
Research role (completeness). Extensionality in general models; relevant to the choice of property domain. *Access: B*.
- [111] Asta Halkjær From, Alexander Birch Jensen, Anders Schlichtkrull, Jørgen Villadsen. [A Sequent Calculus for First-Order Logic](#). *Archive of Formal Proofs* (July 18, 2019).
Research role (completeness). First-order sequent soundness and completeness; comparison of proof presentations. *Access: A*.
- [112] Asta Halkjær From, Frederik Krogsdal Jacobsen. [A Sequent Calculus Prover for First-Order Logic with Functions](#). *Archive of Formal Proofs* (January 31, 2022).
Research role (completeness). Certified sequent proof search with functions; useful for independently encoded first-order fragments. *Access: A*.
- [113] Asta Halkjær From, Jonathan Julian Huerta y Munive, Anders Schlichtkrull. [Completeness of the Q0 Higher-Order Logic](#). *Archive of Formal Proofs* (August 3, 2026).
Research role (completeness). General-model completeness for Q0; supports a precise rather than unrestricted completeness claim. *Access: A*.
- [114] Asta Halkjær From, Jørgen Villadsen. [Soundness and Completeness of Implicational Logic](#). *Archive of Formal Proofs* (September 13, 2022).
Research role (completeness). Soundness and completeness of implication; background for the propositional skeleton of the proof. *Access: A*.
- [115] Asta Halkjær From. [A Naive Prover for First-Order Logic](#). *Archive of Formal Proofs* (March 22, 2022).
Research role (completeness). A certified first-order prover; comparison route after an explicit first-order encoding. *Access: A*.
- [116] Asta Halkjær From. [Soundness and Completeness of an Axiomatic System for First-Order Logic](#). *Archive of Formal Proofs* (September 24, 2021).

- Research role (completeness)*. First-order soundness and completeness; background for proof calculus versus semantics. *Access: A*.
- [117] Asta Halkjær From. [Synthetic Completeness](#). Archive of Formal Proofs (January 9, 2023).
Research role (completeness). Maximal consistent witnessed sets; supports the model-existence step in general completeness arguments. *Access: A*.
 - [118] Cocchiarella, N.B.: [A completeness theorem in second order modal logic](#). Theoria 35 (2), 81–103 (1969).
Research role (completeness). Second-order modal completeness under specified semantics; relevant to quantification over properties. *Access: B*.
 - [119] Fabián Fernando Serran Suárez, Thaynara Arielly de Lima, Mauricio Ayala-Rincón. [Compactness Theorem for Propositional Logic and Combinatorial Applications](#). Archive of Formal Proofs (August 19, 2024).
Research role (completeness). Formal propositional compactness; supports the finite-consistency discussion. *Access: A*.
 - [120] Henkin, L.: [Completeness in the theory of types](#). J. Symb. Log. 15 (2), 81–91 (1950).
Research role (completeness). Henkin completeness for type theory; distinguishes general-model completeness from full semantics. *Access: B*.
 - [121] James Margetson, Tom Ridge. [Completeness theorem](#). Archive of Formal Proofs (September 20, 2004).
Research role (completeness). Formal first-order completeness; foundational contrast with full second-order semantic validity. *Access: A*.
 - [122] Jasmin Christian Blanchette, Andrei Popescu, Dmitriy Traytel. [Abstract Completeness](#). Archive of Formal Proofs (April 16, 2014).
Research role (completeness). Abstract model-existence and completeness machinery for derivation trees. *Access: A*.
 - [123] Jasmin Christian Blanchette, Andrei Popescu, Dmitriy Traytel. [Abstract Soundness](#). Archive of Formal Proofs (February 10, 2017).
Research role (completeness). Abstract soundness of derivation systems; supports external validation of proof rules. *Access: A*.
 - [124] Julius Michaelis, Tobias Nipkow. [Propositional Proof Systems](#). Archive of Formal Proofs (June 21, 2017).
Research role (completeness). Relations among sequent, natural-deduction, Hilbert and resolution presentations. *Access: A*.
 - [125] Leon Henkin. [The completeness of the first-order functional calculus](#). Journal of Symbolic Logic 14, 159–166 (1949).
Research role (completeness). First-order completeness by model construction; supports the compactness and witness discussion. *Access: B*.
 - [126] Matthew Doty. [Class-based Classical Propositional Logic](#). Archive of Formal Proofs (December 15, 2022).
Research role (completeness). Abstract propositional proof systems; foundation for the truth-functional implication used in the manuscript. *Access: A*.
 - [127] Sophie Turret, Lawrence C. Paulson. [Compactness Theorem for First-Order Logic](#). Archive of Formal Proofs (February 26, 2025).
Research role (completeness). Formal first-order compactness; used to distinguish finite satisfiability from realization in a fixed model. *Access: A*.
 - [128] Stefan Berghofer, Asta Halkjær From. [First-Order Logic According to Fitting](#). Archive of Formal Proofs (August 2, 2007).
Research role (completeness). Formal first-order tableaux; comparison with the manuscript’s semantic proof. *Access: A*.
 - [129] Tom Ridge. [A Mechanically Verified, Efficient, Sound and Complete Theorem Prover For First Order Logic](#). Archive of Formal Proofs (September 28, 2004).
Research role (completeness). A verified sound and complete first-order prover; comparison implementation. *Access: A*.
 - [130] Andrei Popescu, Dmitriy Traytel. [An Abstract Formalization of Gödel’s Incompleteness Theorems](#). Archive of Formal Proofs (September 16, 2020).
Research role (incompleteness). Abstract incompleteness hypotheses, including Rosser-style variants; central proof-theoretic reference. *Access: A*.
 - [131] Andrei Popescu, Dmitriy Traytel. [From Abstract to Concrete Gödel’s Incompleteness Theorems—Part I. Archive of Formal Proofs \(September 16, 2020\)..](#)
Research role (incompleteness). Concrete instantiation of abstract incompleteness theorems; distinguishes soundness assumptions from consistency alone. *Access: A*.

- [132] Andrei Popescu, Dmitriy Traytel. [From Abstract to Concrete Gödel’s Incompleteness Theorems—Part II](#). Archive of Formal Proofs (September 16, 2020).
Research role (incompleteness). Concrete validation of abstract second-incompleteness conditions using finite-set theories. *Access: A*.
- [133] Andrei Popescu, Dmitriy Traytel. [Robinson Arithmetic](#). Archive of Formal Proofs (September 16, 2020).
Research role (incompleteness). Robinson arithmetic as an explicit weak arithmetic base for metatheorems. *Access: A*.
- [134] Barkley Rosser. [Extensions of some theorems of Gödel and Church](#). Journal of Symbolic Logic 1, 87-91 (1936).
Research role (incompleteness). Original strengthening of incompleteness under consistency; supports the exact hypothesis used in the paper. *Access: B*.
- [135] Bertram Felgenhauer. [Implementing the Goodstein Function in \$\lambda\$ -Calculus](#). Archive of Formal Proofs (February 21, 2020).
Research role (incompleteness). Goodstein computation in lambda calculus; separates executable definitions from the strength needed to prove universal termination. *Access: A*.
- [136] Emmanuel Gunther, Miguel Pagano, Pedro Sánchez Terraf. [Formalization of Forcing in Isabelle/ZF](#). Archive of Formal Proofs (May 6, 2020).
Research role (incompleteness). Generic model extensions under explicit set-theoretic assumptions; comparison for relative model arguments. *Access: A*.
- [137] Gödel, K.: [Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme i. Monatshefte für Mathematik und Physik 38 , 173–198 \(1931\)](#).
Research role (incompleteness). Original incompleteness results; constrains self-certification in effective arithmetic theories. *Access: B*.
- [138] H. Friedman. [A Divine Consistency Proof for Mathematics](#). In Ontology of Divinity, 645-696. De Gruyter (2024).
Research role (incompleteness). Conditional consistency argument with additional divine assumptions; relevant to the limits of internal consistency claims. *Access: B*.
- [139] Lawrence C. Paulson. [Gödel’s Incompleteness Theorems](#). Archive of Formal Proofs (November 17, 2013).
Research role (incompleteness). Formal incompleteness and Lob results; central reference for limits of internal consistency certification. *Access: A*.
- [140] Lawrence C. Paulson. [The Hereditarily Finite Sets](#). Archive of Formal Proofs (November 17, 2013).
Research role (incompleteness). Hereditarily finite sets as an arithmetic-coding foundation. *Access: A*.
- [141] Lawrence C. Paulson. [Zermelo Fraenkel Set Theory in Higher-Order Logic](#). Archive of Formal Proofs (October 24, 2019).
Research role (incompleteness). Set theory within HOL; clarifies the additional foundations behind large semantic model classes. *Access: A*.
- [142] Mühlenbeck, C.: [Der Zusammenhang zwischen Gödels Unvollständigkeitssätzen, der Kontinuumshypothese und dem Ontologischen Gottesbeweis und seine Auswirkungen auf die metamathematische Grundlegung der Mathematik](#). Bachelorarbeit, Freie Universität Berlin (2024).
Research role (incompleteness). Research comparison of incompleteness, the continuum hypothesis, and ontological formalization; contextual, not a new premise. *Access: B*.
- [143] Studtmann, P.: [God and the numbers](#). J. Philos. 120 , 641–655 (2023).
Research role (incompleteness). Examines relations between God-language and arithmetic; comparison rather than an inference from arithmetic alone. *Access: B*.
- [144] Štěpán Holub, Zuzana Haniková. [Axiomatic theory of hereditarily finite sets and its fragments](#). Archive of Formal Proofs (July 3, 2026).
Research role (incompleteness). Finite-set axiom fragments; relevant to the arithmetic strength needed for incompleteness. *Access: A*.
- [145] Adnan Mohammed Ahmed, Balazs Toth. [Typed Ordered Resolution](#). Archive of Formal Proofs (June 11, 2025).
Research role (verification). Typed ordered resolution; supports type-preserving automated proof search. *Access: A*.
- [146] Anders Schlichtkrull, Jasmin Christian Blanchette, Dmitriy Traytel, Uwe Waldmann. [Formalization of Bachmair and Ganzinger’s Ordered Resolution Prover](#). Archive of Formal Proofs (January 18, 2018).
Research role (verification). Ordered-resolution metatheory; relevant to certified automated inference. *Access: A*.

- [147] Anders Schlichtkrull, Jasmin Christian Blanchette, Dmitriy Traytel. [A Verified Functional Implementation of Bachmair and Ganzinger’s Ordered Resolution Prover](#). Archive of Formal Proofs (November 23, 2018). *Research role (verification)*. Verified implementation of ordered resolution; distinguishes abstract calculi from executable provers. *Access: A*.
- [148] Anders Schlichtkrull. [The Resolution Calculus for First-Order Logic](#). Archive of Formal Proofs (June 30, 2016). *Research role (verification)*. Resolution soundness and completeness; background for clause-based independent checking. *Access: A*.
- [149] Andrews, P.B., Bishop, M., Issar, S., Nesmith, D., Pfenning, F., Xi, H.: [Tps: a theorem-proving system for classical type theory](#). J. Autom. Reason. 16 , 321–353 (1996). *Research role (verification)*. Classical type-theory theorem prover; historical implementation of higher-order proof search. *Access: B*.
- [150] Andrews, P.B., Brown, C.E.: [Tps: a hybrid automatic-interactive system for developing proofs](#). J. Appl. Log. 4 (4), 367–395 (2006). *Research role (verification)*. Hybrid interactive and automated type-theoretic proof development. *Access: B*.
- [151] Andrews, P.B.: [Some reflections on proof transformations](#). In: Hutter, D., Stephan, W. (eds.) Mechanizing Mathematical Reasoning: Essays in Honor of Jörg H. Siekmann on the Occasion of His 60th Birthday, pp. 14–29. Springer, Berlin (2005). *Research role (verification)*. Proof transformations; relevant to moving between readable derivations and formal certificates. *Access: B*.
- [152] Arthur Freitas Ramos, David Barros Hulak, Ruy Jose Guerra Barretto de Queiroz. [A Reusable Isabelle/HOL Framework for Propositional Labelled Natural Deduction](#). Archive of Formal Proofs (May 27, 2026). *Research role (verification)*. Structural metatheory of labelled deduction; useful for representing modal proof steps. *Access: A*.
- [153] Avigad, J.: [Automated reasoning for mathematics](#). In: Benzmüller, C., Heule, M.J.H., Schmidt, R.A. (eds.) Automated Reasoning—12th International Joint Conference, IJCAR 2024, Nancy, France, July 3–6, 2024, Proceedings, Part I. Lecture Notes in Computer Science, vol. 14739, pp. 3–20. Springer, Cham (2024). *Research role (verification)*. Survey of automated reasoning for mathematics; distinguishes verified results from search output. *Access: B*.
- [154] Barbosa, H., Barrett, C.W., Brain, M., Kremer, G., Lachnitt, H., Mann, M., Mohamed, A., Mohamed, M., Niemetz, A., Nötzli, A., Ozdemir, A., Preiner, M., Reynolds, A., Sheng, Y., Tinelli, C., Zohar, Y.: [cvc5: a versatile and industrial-strength SMT solver](#). In: Fisman, D., Rosu, G. (eds.) Tools and Algorithms for the Construction and Analysis of Systems—28th International Conference, TACAS 2022, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2022, Munich, Germany, April 2–7, 2022, Proceedings, Part I. Lecture Notes in Computer Science, vol. 13243, pp. 415–442. Springer, Cham (2022). *Research role (verification)*. SMT solver architecture; relevant to finite checking and proof reconstruction infrastructure. *Access: B*.
- [155] Bentkamp, A., Blanchette, J., Nummelin, V., Tourret, S., Vukmirovic, P., Waldmann, U.: [Mechanical mathematicians](#). Commun. ACM 66 (4), 80–90 (2023). *Research role (verification)*. Survey background on proof assistants and automated deductions; distinguishes proof search from certification. *Access: B*.
- [156] Bentkamp, A., Blanchette, J., Tourret, S., Vukmirovic, P.: [Superposition for higher-order logic](#). J. Autom. Reason. 67 (1), 10 (2023). *Research role (verification)*. Higher-order superposition; relevant proof-search machinery for intensional encodings. *Access: B*.
- [157] Benzmüller, C., Miller, D.: [Automation of higher-order logic](#). In: Gabbay, D.M., Siekmann, J.H., Woods, J. (eds.) Handbook of the History of Logic. Computational Logic, vol. 9, pp. 215–254. Elsevier, North Holland (2014). *Research role (verification)*. Background on higher-order automated reasoning needed for quantification over properties. *Access: B*.
- [158] Benzmüller, C., Sultana, N., Paulson, L.C., Theiß, F.: [The higher-order prover LEO-II](#). J. Autom. Reason. 55 (4), 389–404 (2015). *Research role (verification)*. Higher-order prover used in research on the ontological argument and its inconsistency. *Access: B*.
- [159] Bertot, Y., Castéran, P.: [Interactive Theorem Proving and Program Development-Coq’Art: The Calculus of Inductive Constructions](#). Texts in Theoretical Computer Science. An EATCS Series, Springer, Berlin (2004).

- Research role (verification).* Coq proof construction and inductive foundations; comparison proof-assistant architecture. *Access:* B.
- [160] Bhayat, A., Suda, M.: [A higher-order vampire \(short paper\)](#). In: Benz Müller, C., Heule, M.J.H., Schmidt, R.A. (eds.) Automated Reasoning—12th International Joint Conference, IJCAR 2024, Nancy, France, July 3–6, 2024, Proceedings, Part I. Lecture Notes in Computer Science, vol. 14739, pp. 75–85. Springer, Cham (2024).
Research role (verification). Higher-order automated prover implementation; alternative search backend. *Access:* B.
- [161] Blanchette, J.C., Böhme, S., Paulson, L.C.: [Extending Sledgehammer with SMT solvers](#). J. Autom. Reason. 51 (1), 109–128 (2013).
Research role (verification). Integration of external solvers with Isabelle; relevant to reconstructing rather than trusting solver answers. *Access:* B.
- [162] Blanchette, J.C., Nipkow, T.: [Nitpick: A counterexample generator for higher-order logic based on a relational model finder](#). In: Kaufmann, M., Paulson, L.C. (eds.) Interactive Theorem Proving—ITP 2010. LNCS, vol. 6172, pp. 131–146. Springer, Berlin (2010).
Research role (verification). Finite countermodel generation for higher-order logic; motivates explicit finite-model checks. *Access:* B.
- [163] Brown, C.E.: [Satallax: an automatic higher-order prover](#). In: Gramlich, B., Miller, D., Sattler, U. (eds.) Automated Reasoning—6th International Joint Conference, IJCAR 2012, Manchester, UK, June 26–29, 2012. Proceedings. Lecture Notes in Computer Science, vol. 7364, pp. 111–117. Springer, Berlin (2012).
Research role (verification). Higher-order tableau prover; published automation route for typed logical problems. *Access:* B.
- [164] Dechesne, F., Nederpelt, R.P.: [N.g. de bruijn \(1918–2012\) and his road to automath, the earliest proof checker](#). Math. Intell. 34, 4–11 (2012).
Research role (verification). Historical proof-checker design; motivates separating discovery from trusted checking. *Access:* B.
- [165] Dmitriy Traytel. [Derivatives of Logical Formulas](#). Archive of Formal Proofs (May 28, 2015).
Research role (verification). Decision procedures for restricted quantified logics; separates decidable fragments from full higher-order validity. *Access:* A.
- [166] Emmanuel Gunther, Miguel Pagano, Pedro Sánchez Terraf, Matías Steinberg. [Transitive Models of Fragments of ZFC](#). Archive of Formal Proofs (March 3, 2022).
Research role (verification). Transitive set models; illustrates why stronger model assumptions must be stated. *Access:* A.
- [167] Filip Smola, Jacques D. Fleuriot. [Deep Embedding of Intuitionistic Linear Logic](#). Archive of Formal Proofs (November 25, 2024).
Research role (verification). A different proof calculus; clarifies why changing structural rules changes the logical setting. *Access:* A.
- [168] Harrison, J.: [HOL light: An overview](#). In: Berghofer, S., Nipkow, T., Urban, C., Wenzel, M. (eds.) Theorem Proving in Higher Order Logics, 22nd International Conference, TPHOLs 2009, Munich, Germany, August 17–20, 2009. Proceedings. Lecture Notes in Computer Science, vol. 5674, pp. 60–66. Springer, Berlin (2009).
Research role (verification). HOL Light architecture as a comparison implementation with a small logical kernel. *Access:* B.
- [169] J. A. Robinson. [A Machine-Oriented Logic Based on the Resolution Principle](#). Journal of the ACM 12, 23–41 (1965).
Research role (verification). Original resolution principle; foundational source for clause-based verification infrastructure. *Access:* B.
- [170] Jasmin Christian Blanchette, Andrei Popescu. [Sound and Complete Sort Encodings for First-Order Logic](#). Archive of Formal Proofs (June 27, 2013).
Research role (verification). Sound sort encodings; guards against losing typing information in automated translations. *Access:* A.
- [171] Joachim Breitner, Denis Lohner. [The meta theory of the Incredible Proof Machine](#). Archive of Formal Proofs (May 20, 2016).
Research role (verification). A proof-graph representation and its metatheory; comparison with linear written derivations. *Access:* A.
- [172] Kaliszyk, C., Urban, J.: [Hol\(y\)hammer: online ATP service for HOL light](#). Math. Comput. Sci. 9 (1), 5–22 (2015).

- Research role (verification)*. Automated proof assistance for HOL Light; comparison of certification pipelines. *Access: B*.
- [173] Kevin Kappelmann. [Unification Utilities for Isabelle/ML](#). Archive of Formal Proofs (September 19, 2023). *Research role (verification)*. Proof-producing unification; relevant to checked instantiation of quantified properties. *Access: A*.
- [174] Kevin Kappelmann. [Zippy - Generic White-Box Proof Search with Zippers](#). Archive of Formal Proofs (November 11, 2025). *Research role (verification)*. Proof-search orchestration; aids discovery without becoming a new axiom or proof certificate. *Access: A*.
- [175] Lawrence C. Paulson. [Finite Automata in Hereditarily Finite Set Theory](#). Archive of Formal Proofs (February 5, 2015). *Research role (verification)*. Finite-set representations of syntax and automata; supporting infrastructure for arithmetized and encoded reasoning. *Access: A*.
- [176] Martin Desharnais-Schäfer. A Formalization of the SCL(FOL) Calculus: [Simple Clause Learning for First-Order Logic](#). Archive of Formal Proofs (April 20, 2023). *Research role (verification)*. Clause-learning soundness and completeness; alternative proof-search infrastructure. *Access: A*.
- [177] Martin Desharnais-Schäfer. [SCL Simulates Nonredundant Ground Resolution](#). Archive of Formal Proofs (October 31, 2024). *Research role (verification)*. Relation between clause learning and ordered resolution; relevant to automated derivation search. *Access: A*.
- [178] Moura, L., Ullrich, S.: [The Lean 4 theorem prover and programming language](#). In: Platzer, A., Sutcliffe, G. (eds.) Automated Deduction—CADE 28—28th International Conference on Automated Deduction, Virtual Event, July 12–15, 2021, Proceedings. Lecture Notes in Computer Science, vol. 12699, pp. 625–635. Springer, Cham (2021). *Research role (verification)*. Lean architecture; relevant to independent reimplementations, not evidence that a Lean proof was run here. *Access: B*.
- [179] Nicolas Peltier. [A Variant of the Superposition Calculus](#). Archive of Formal Proofs (September 6, 2016). *Research role (verification)*. Superposition soundness and completeness; supports equality-aware proof-search methods. *Access: A*.
- [180] Nicolas Peltier. [Propositional Resolution and Prime Implicates Generation](#). Archive of Formal Proofs (March 11, 2016). *Research role (verification)*. Propositional resolution and prime implicates; aids checking the Boolean skeleton of assumptions. *Access: A*.
- [181] Niederhauser, J., Brown, C.E., Kaliszky, C.: [Tableaux for automated reasoning in dependently-typed higher-order logic](#). In: Benzmüller, C., Heule, M.J.H., Schmidt, R.A. (eds.) Automated Reasoning—12th International Joint Conference, IJCAR 2024, Nancy, France, July 3–6, 2024, Proceedings, Part I. Lecture Notes in Computer Science, vol. 14739, pp. 86–104. Springer, Cham (2024). *Research role (verification)*. Tableau automation for dependent higher-order logic; background for proof-search alternatives. *Access: B*.
- [182] Nipkow, T., Klein, G.: [Concrete Semantics-With Isabelle/HOL](#). Springer, Cham (2014). *Research role (verification)*. Structured semantic proofs in Isabelle; methodological background for explicit model definitions. *Access: B*.
- [183] Nipkow, T., Paulson, L.C., Wenzel, M.: [Isabelle/HOL—A Proof Assistant for Higher-Order Logic](#). LNCS, vol. 2283. Springer, Berlin (2002). *Research role (verification)*. Reference for the Isabelle/HOL environment used in the published formalizations. *Access: B*.
- [184] Owre, S., Rushby, J.M., Shankar, N.: [PVS: A prototype verification system](#). In: Kapur, D. (ed.) Automated Deduction—CADE-11, 11th International Conference on Automated Deduction, Saratoga Springs, NY, USA, June 15–18, 1992, Proceedings. Lecture Notes in Computer Science, vol. 607, pp. 748–752. Springer, Berlin (1992). *Research role (verification)*. PVS verification architecture; background on alternative typed proof environments. *Access: B*.
- [185] Peter Chapman. [Invertibility in Sequent Calculi](#). Archive of Formal Proofs (August 28, 2009). *Research role (verification)*. Invertible sequent rules; relevant to search without changing provability. *Access: A*.

- [186] Schulz, S., Cruanes, S., Vukmirovic, P.: [Faster, higher, stronger: E 2.3](#). In: Fontaine, P. (ed.) Automated Deduction—CADE 27—27th International Conference on Automated Deduction, Natal, Brazil, August 27–30, 2019, Proceedings. Lecture Notes in Computer Science, vol. 11716, pp. 495–507. Springer, Cham (2019).
Research role (verification). First-order prover implementation; relevant after a justified translation to first-order clauses. *Access: B*.
- [187] Slind, K., Norrish, M.: [A brief overview of HOL4](#). In: Mohamed, O.A., Muñoz, C.A., Tahar, S. (eds.) Theorem Proving in Higher Order Logics, 21st International Conference, TPHOLs 2008, Montreal, Canada, August 18–21, 2008. Proceedings. Lecture Notes in Computer Science, vol. 5170, pp. 28–32. Springer, Berlin (2008).
Research role (verification). HOL4 architecture as a comparison implementation of higher-order reasoning. *Access: B*.
- [188] Steen, A., Benzmüller, C.: [Extensional higher-order paramodulation in Leo-III](#). J. Autom. Reason. 65 (6), 775–807 (2021).
Research role (verification). Higher-order paramodulation and extensionality; relevant to equality-aware property reasoning. *Access: B*.
- [189] Stefan Berghofer, Markus Reiter. [Formalizing the Logic-Automaton Connection](#). Archive of Formal Proofs (December 3, 2009).
Research role (verification). Logic-automaton connection; supporting verification of restricted arithmetic fragments. *Access: A*.
- [190] Sutcliffe, G., Benzmüller, C.: [Automated reasoning in higher-order logic using the TPTP THF infrastructure](#). J. Formal. Reason. 3 (1), 1–27 (2010).
Research role (verification). Typed higher-order problem interchange; relevant to reproducible encodings of the axioms. *Access: B*.
- [191] Sutcliffe, G.: [Stepping stones in the TPTP world](#). In: Benzmüller, C., Heule, M.J.H., Schmidt, R.A. (eds.) Automated Reasoning—12th International Joint Conference, IJCAR 2024, Nancy, France, July 3–6, 2024, Proceedings, Part I. Lecture Notes in Computer Science, vol. 14739, pp. 30–50. Springer, Cham (2024).
Research role (verification). Logical benchmark infrastructure and representation conventions; supporting reproducibility literature. *Access: B*.
- [192] Sutcliffe, G.: [The CADE ATP system competition—CASC](#). AI Mag. 37 (2), 99–101 (2016).
Research role (verification). Competition methodology; solver success is distinct from the truth of the encoded assumptions. *Access: B*.
- [193] Sutcliffe, G.: [The TPTP problem library and associated infrastructure—from CNF to TH0, TPTP v6.4.0](#). J. Autom. Reason. 59 (4), 483–502 (2017).
Research role (verification). Benchmark infrastructure for comparing theorem provers on precisely stated problems. *Access: B*.
- [194] Tobias Nipkow. [Boolean Expression Checkers](#). Archive of Formal Proofs (June 8, 2014).
Research role (verification). Executable Boolean satisfiability and equivalence checks; supports finite validation of truth-functional steps. *Access: A*.
- [195] Tobias Nipkow. [Quantifier Elimination for Linear Arithmetic](#). Archive of Formal Proofs (January 11, 2008).
Research role (verification). Verified arithmetic quantifier elimination; contrasts algorithmic fragments with incompleteness hypotheses. *Access: A*.
- [196] Wenzel, M.: [Isabelle, Isar—a versatile environment for human readable formal proof documents](#). PhD thesis, Technical University Munich, Germany (2002).
Research role (verification). Readable structured proof documents; methodological reference for lemma dependencies. *Access: B*.
- [197] Yutaka Nagashima. [Proof Strategy Language](#). Archive of Formal Proofs (December 20, 2016).
Research role (verification). Proof strategy automation; separates finding proofs from validating them. *Access: A*.
- [198] Manuel Eberl, Johannes Hölzl, Tobias Nipkow. [A Verified Compiler for Probability Density Functions](#). Archive of Formal Proofs (October 9, 2014).
Research role (probability). Certified construction of probability densities; contrast with an unspecified prior over models. *Access: A*.
- [199] Matthew Doty. [A Sound and Complete Calculus for Probability Inequalities](#). Archive of Formal Proofs (February 20, 2023).
Research role (probability). Probability-inequality calculus; directly relates logical validity to universal probability bounds. *Access: A*.

- [200] Matthew Doty. [Suppes' Theorem For Probability Logic](#). Archive of Formal Proofs (January 22, 2023).
Research role (probability). Suppes-style probability logic; directly motivates the entailment-to-probability bound proved in the paper. *Access: A*.
- [201] Michikazu Hirata, Yasuhiko Minamide. [S-Finite Measure Monad on Quasi-Borel Spaces](#). Archive of Formal Proofs (August 8, 2023).
Research role (probability). Higher-order probability with conditioning; relevant when formalizing a measure over semantic objects. *Access: A*.
- [202] Michikazu Hirata. [Coproduct Measure](#). Archive of Formal Proofs (June 4, 2024).
Research role (probability). Constructing measures on disjoint model spaces; relevant to combining explicitly specified model families. *Access: A*.